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LETTER TO THE EDITOR

Theoretical study of vortex lines in a spatially modulated Josephson junction

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Abstract. The energy and oscillation frequency of a single vortex line or soliton in a long Josephson junction with a spatially modulated Josephson penetration depth are calculated systematically. The resulting critical magnetic field H_{c1}, and the stability and pinning of the different soliton solutions are discussed. The pinning forces are strongest if the period length of the modulation is about three times larger than the average penetration depth (or coherence length) of the junction, although the absolute barrier height against free vortex movement still increases.

The magnetic properties of type II superconductors and Josephson junctions (see e.g. Saint-James *et al* 1969 for a review) are strongly determined by the presence of vortex lines containing one unit of the magnetic flux quantum. The pinning of these vortex lines in spatially inhomogeneous superconductors has an important influence on their magnetic properties, such as for example a magnetic hysteresis due to 'frozen-in' magnetic flux. The dissipation of energy due to the movement of vortex lines at high currents is reduced if the vortex lines are pinned at defects. The new high-T_c superconductors consist essentially of two-dimensional superconducting layers at a distance apart which is of the order of the superconducting coherence length (perpendicular to the planes). Thus, they have an intrinsically modulated structure that might have an important influence on the properties of vortex lines, especially their mobility. In principle, spatially inhomogeneous superconducting structures could be used to store information. The presence or absence of a vortex line at a well defined site would correspond to one bit of information.

But, in spite of all this interest, it is only possible in special cases to treat the pinning of vortex lines and vortex-line lattices in a quantitatively accurate way (Saint-James *et al* 1969, Stampfli and Rice 1986). Usually one has to use an approximate and often incomplete analytical or numerical treatment. One instance, where rather exact numerical work is possible, is the case of spatially modulated Josephson junctions, due to their essentially one-dimensional nature. Kulić (1987) has previously proposed to study this problem, but he has only given few results. In this Letter we intend to study the pinning of vortex lines or solitons in periodically modulated Josephson junctions in more detail. The free energies of solitons and the corresponding critical magnetic fields for flux penetration are given, together with the absolute barrier against free-soliton movement and pinning-force constants.

To set up our model of the modulated junction we assume that the critical current density J_1 of the Josephson junction is constant in space and that the Josephson penetration depth λ_J (or coherence length) is spatially modulated. This could be achieved with an inhomogeneous doping of the superconducting layers of the junction with non-magnetic scatterers (Kulić 1987, Saint-James *et al* 1969), resulting in spatially varying electronic mean free paths. The junction is assumed to be long in the x direction and narrow in the y direction. It is oriented parallel to the xy plane and modulated depending on the x coordinate. Magnetic fields are applied parallel to the y direction. We neglect boundary effects due to the finite length of the junction in the x direction. The free energy density per unit area is then (Barone and Paterno 1982)

$$f(x) = (\hbar J_1/2e)[(1 - \cos\varphi(x)) + \tfrac{1}{2}\lambda_J^2(x)(\mathrm{d}\varphi(x)/\mathrm{d}x)^2] \tag{1}$$

where $\varphi(x)$ is the difference in the phase of the superconducting order parameter in the superconductors at both sides of the junction. The free energy (for a unit sample thickness in y direction) is $F = \int f\,\mathrm{d}x$. The modulated Josephson penetration depth is $\lambda_J^2(x) = \hbar c^2/8\pi e J_1 d(x)$ where $c = \varepsilon/4\pi t$ is the capacity per unit area of the dielectric layer of thickness t and dielectric constant ε. For non-magnetic scattering $d(x) = \lambda_1(x) + \lambda_2(x) + t = d_0 g(x)$, where $\lambda_1(x)$ and $\lambda_2(x)$ are the spatially modulated mean free paths of the superconducting layers on both sides of the junction. Then we can put $\lambda_J^2(x) = \lambda_0^2/g(x)$ and $\lambda_0^2 = \hbar c^2/8\pi J_1 d_0$, where J_1 is the critical current density of the junction. For simplicity we assume a sinusoidal modulation $g(x) = 1 + a\cos(2\pi x/\Lambda)$, where a is the amplitude of modulation. Other steeper or asymmetric profiles could be more effective for pinning vortex lines, but the conclusions of our work should also apply in these cases, at least qualitatively.

It is convenient to introduce dimensionless units in order to eliminate all irrelevant parameters. The transformation into the original units is defined by

$$x = \lambda_0 u \tag{2a}$$

for positions and

$$F = (\hbar J_1 \lambda_0/2e)\tilde{F} \tag{2b}$$

for the free energy. Note that $\tilde{F} = \int \tilde{f}\,\mathrm{d}u$ and thus the free energy density is in the new units

$$\tilde{f}(u) = 1 - \cos\varphi + \tfrac{1}{2}g^{-1}(u)(\mathrm{d}\varphi/\mathrm{d}u)^2 \tag{3a}$$

with

$$g(u) = 1 + a\cos(2\pi u/L) \tag{3b}$$

and $L = \Lambda/\lambda_0$. The relevant parameters, which define the modulated Josephson junction, are thus the amplitude a of the modulation and the ratio L of the length Λ of the period of the modulation $g(u)$ to the Josephson penetration depth λ_0 of the uniform barrier ($a = 0$).

We want to study a single soliton φ_s in such a modulated Josephson junction, and to consider its free energy. From the variation of $\tilde{F}$ with respect to φ (or using Lagrange–Euler equations, Barone and Paterno 1982) we obtain a modulated Sine–Gordon equation for φ

$$(\mathrm{d}/\mathrm{d}u)[(1/g(u))(\mathrm{d}\varphi/\mathrm{d}u)] = \sin\varphi. \tag{4}$$

The appropriate boundary conditions are $\varphi_s(+\infty) = 2\pi$ and $\varphi_s(-\infty) = 0$. Such a vortex line becomes energetically stable for sufficiently large external magnetic fields $H_0 > H_{c1}$ and the penetration of vortex lines into the junction becomes possible (Kulik 1967). The critical field H_{c1} is obtained from

$$H_{c1} = (4\pi/\Phi_0)F_s = (4\pi/\Phi_0)(\hbar J_1\lambda_0/2e)\tilde{F}_s \tag{5}$$

where $\Phi_0 = hc/2e$ is the magnetic flux quantum and $\tilde{F}_s$ is the free energy of the soliton φ_s. Because of the spatially varying $g(u)$, there are two different single soliton solutions of (4) with different free energies, depending whether the maximum of $(\mathrm{d}\varphi/\mathrm{d}u)^2$ lies around the maximum or minimum of $g(u)$. Note that our particular choice of $g(u)$ is symmetric with respect to a change of the sign of a (together with a translation by $L/2$) and that $g(-u) = g(u)$. Thus we obtain both soliton solutions with the same boundary conditions $\varphi_s(0) = \pi$ and $\varphi_s(-\infty) = 0$, simply by changing the sign of a. The maximum of $(\mathrm{d}\varphi/\mathrm{d}u)^2$ lies at $u = 0$ and the soliton of low (or high) free energy is obtained using a positive (or negative) a.

It is interesting to consider the limiting cases for L, because we can obtain analytical results. In the limit $L \to \infty$ it is $g(u) \sim 1 + a =$ constant in the region where $\mathrm{d}\varphi_s/\mathrm{d}u \neq 0$ and the solution of (4) is therefore (Kulik 1967)

$$\varphi_s(u) \simeq 4\tan^{-1}\exp[(1+a)^{1/2}u] \qquad L \gg (1+a)^{-1/2}$$

and the free energy of the soliton is

$$\tilde{F}_s = 8/(1+a)^{1/2}. \tag{6}$$

Note that for small a we have $\tilde{F}_s \sim 8(1 - a/2)$, which is essentially equivalent to the result given by Kulić (1987) in his equation (5). In the limit $L \to 0$ we obtain from the soliton $\varphi_0(u) = 4\tan^{-1}\exp(u)$, of the homogeneous junction ($a = 0$), an approximate solution of (4)

$$\varphi_s(u) \simeq \varphi_0(u) + (La/2\pi)\sin[(2\pi/L)u](\mathrm{d}/\mathrm{d}u)\varphi_0(u). \tag{7}$$

It is easy to verify that (4) is thus fulfilled up to terms of order $|La|$ and that the free energy is

$$\tilde{F}_s(a, L) - 8 = \text{constant} \times (La)^2 \qquad \text{for } L \to 0. \tag{8}$$

Thus we obtain that the free energy of the solitons become the same as in the homogeneous junction ($a = 0$), $\tilde{F}_s \to 8$, for $L \to 0$.

In general, (4) can only be solved numerically because 'inverse scattering methods' cannot be used since the continuous-translation symmetry is broken by $g(u)$. Note that the asymptotic behaviour given by equations (6)–(8) differs widely for $L \to 0$ and $L \to \infty$; thus there is probably no simple analytic approximation for φ_s and F_s for finite L and a. From the symmetry of $g(u)$ follows that $\varphi_s(u) = 2\pi - \varphi_s(-u)$ and $\tilde{f}(u) = \tilde{f}(-u)$ if the boundary conditions $\varphi_s(-\infty) = 0$ and $\varphi_s(0) = \pi$ are applied. To make certain that no soliton–antisoliton pairs appear in the numerical solution we have to check that $\mathrm{d}\varphi_s/\mathrm{d}u > 0$ for all u. The boundary condition $\varphi_s(-\infty) = 0$ cannot be used directly in a numerical calculation and requires a special treatment. Note that if φ is small enough (e.g. for $u \to -\infty$), we can put $\sin\varphi \simeq \varphi$ in (4) and obtain the linear differential equation

$$\varphi - (\mathrm{d}/\mathrm{d}u)[(1/g(u))(\mathrm{d}\varphi/\mathrm{d}u)] = 0 \qquad |\varphi| \ll 1.$$

From Bloch's theorem (Ashcroft and Mermin 1976) we obtain that the elementary solutions of this differential equation are of the form

$$\varphi_\pm(u) = v_\pm(u)\exp(\pm\kappa u) \tag{9}$$

where $v_\pm(u)$ are periodic functions with period length L and κ is an appropriate constant. Then, φ_+ is easy to find using numerical integration and transfer matrix methods. Note that $\varphi_+(-\infty) = 0$ and $\varphi_s \propto \varphi_+$ for $u \to -\infty$. Thus, at a sufficiently large negative u_0, we obtain a new boundary condition $(\mathrm{d}/\mathrm{d}u)\varphi_s(u_0) = \varphi_s(u_0)(\mathrm{d}/\mathrm{d}u)\ln\varphi_+(u_0)$, which is equivalent to $\varphi_s(-\infty) = 0$. We have now to search for an initial value $\varphi_s(u_0)$ which, together with the above condition for $(\mathrm{d}/\mathrm{d}u)\varphi_s(u_0)$, results in $\varphi_s(0) = \pi$ from a numerical integration of (4). This 'shooting method' is treated in detail by Dahlquist and Björg (1974). Thus φ_s is calculated numerically and the free energy $\tilde{F}_s$ is obtained from numerically integrating (3).

To examine the stability of the soliton φ_s we have to consider small time-dependent perturbations and thus $\varphi(u, \tau) = \varphi_s(u) + \psi(u, \tau)$, where $|\psi| \ll 1$ is a small perturbation. It is convenient to introduce dimensionless units for time $\tau = (2eJ_1/c\hbar)^{-1/2}\tilde{\tau}$ and for frequency $\omega = (2eJ_1/c\hbar)^{1/2}\tilde{\omega}$, where c is the capacity per unit area of the junction. The partial differential equation, which determines the dynamical behaviour of the Josephson junction is in these units (Kulik 1967, Fetter and Stephen 1968)

$$(\partial/\partial u)[(1/g(u))(\partial\varphi/\partial u)] - \partial^2\varphi/\partial\tilde{\tau}^2 = \sin\varphi \tag{10}$$

where the dissipative term $\partial^2\varphi/\partial\tilde{\tau}^2$, due to finite voltages across the junction, has been added to (5). Assuming a small perturbation of the form $\psi(u, \tilde{\tau}) = \psi(u)\exp(-\mathrm{i}\tilde{\omega}\tilde{\tau})$, where ω is the vibration frequency of the soliton, we linearise around φ_s and obtain from (10)

$$\omega^2\psi = \psi\cos(\varphi_s) - (\mathrm{d}/\mathrm{d}u)[(1/g(u))(\mathrm{d}\psi/\mathrm{d}u)]. \tag{11}$$

This is a linear eigenvalue problem for ω^2 with the boundary conditions $\psi(\pm\infty) = 0$. From the symmetries of $g(u)$ and $\varphi_s(u)$ it follows that $\psi(u) = \psi(-u)$ and thus $(\mathrm{d}/\mathrm{d}u)\psi(0) = 0$.

For the homogeneous junction ($a = 0$) it is simply $\psi = (\mathrm{d}/\mathrm{d}u)\varphi_s$ and $\omega^2 = 0$; such a perturbation corresponds to a translation of φ_s. Similarly, we obtain that $\omega^2 \to 0$ for $L \to \infty$ (because $g(u) \simeq$ constant where $\psi \neq 0$) and for $L \to 0$ (because $\tilde{F}_s \to 8$ for all solitons). For finite L we have to calculate $\omega^2 \neq 0$ numerically and as for φ_s the boundary condition $\psi(-\infty) = 0$ requires a special treatment. For sufficiently large negative u we can put $\cos(\varphi_s) \simeq 1$ and (11) becomes

$$(1 - \omega^2)\psi - (\mathrm{d}/\mathrm{d}u)[(1/g(u))(\mathrm{d}\psi/\mathrm{d}u)] = 0 \qquad |\varphi| \ll 1.$$

As before, the elementary solutions of this differential equation are of the form given by (9). Note that in this case $\varphi_\pm$ and κ depend on ω^2. Given a trial value for ω^2, we use $\psi(u_0) = 1$ and $(\mathrm{d}/\mathrm{d}u)\psi(u_0) = (\mathrm{d}/\mathrm{d}u)\ln\varphi_+(u_0)$ as starting values for a numerical extrapolation of (11), as discussed for φ_s. The correct value is found if the boundary condition $(\mathrm{d}/\mathrm{d}u)\psi(0) = 0$ is fulfilled.

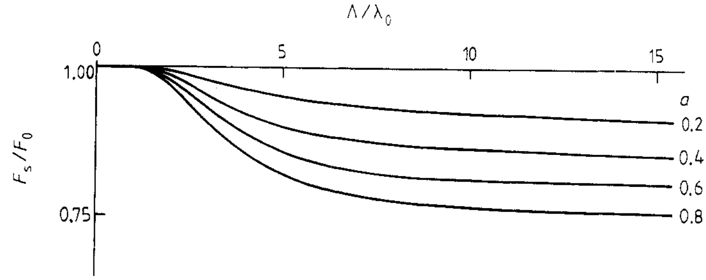

Figure 1. Free energy F_s (per unit sample thickness) of a single vortex line in a spatially modulated Josephson junction. The Josephson penetration depth is $\lambda_J^2(x) = \lambda_0^2/g(x)$, where $g(x) = 1 + a\cos(2\pi x/\Lambda)$. Curves for different values of a, indicated in the figure, are given. $F_0 = 4\hbar J_1\lambda_0/e$ is the free energy of the vortex line in the uniform junction $\lambda_J(x) = \lambda_0$, where J_1 is the critical current density of the junction. Values for the vortex line of low energy are given here.

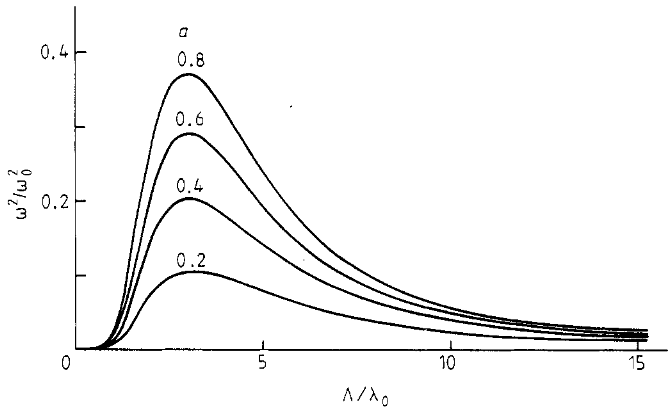

Figure 2. Oscillation frequency ω of the vortex line of low energy in the modulated Josephson junction (see caption of figure 1). Here $\omega_0^2 = 2eJ_1/c\hbar$, where c is the capacity per unit area of the junction. Note that $\omega^2 > 0$ is proportional to a phenomenological pinning force constant.

Results for the free energy F_s and frequency of oscillation ω of a single vortex line or soliton in the modulated Josephson junction (1) are presented in figures 1 to 4. The two different solutions of low or high energy are located around the minimum or maximum of the modulated Josephson penetration depth $\lambda_J(x)$, respectively. Curves for different constant values for the amplitude a of modulation are given as functions of the period of modulation $L = \Lambda/\lambda_0$.

Our numerical results have the correct limiting behaviour given by (6) for $\Lambda \to \infty$ and by (8) for $\Lambda \to 0$. Note that $\omega^2 \to 0$ in both limiting cases. The width of a vortex line is of the order of about $2\lambda_0$, thus if $\Lambda \lesssim 2\lambda_0$ there is practically no difference in the free energy between the two vortex-line solutions and $\omega \simeq 0$. For larger and increasing Λ, the difference between the free energies increases monotonically and (5) defines two critical magnetic fields $H_{c1a} < H_{c1b}$, where H_{c1a} and H_{c1b} are given by F_s in figure 1 and figure 2. For an external field $H_0 > H_{c1a}$, vortex lines can penetrate into the junction, but presumably they are confined to the regions where $\lambda_J(x)$ is small. Because of the pinning of vortex lines it is possible to have frozen-in magnetic flux lines even at smaller

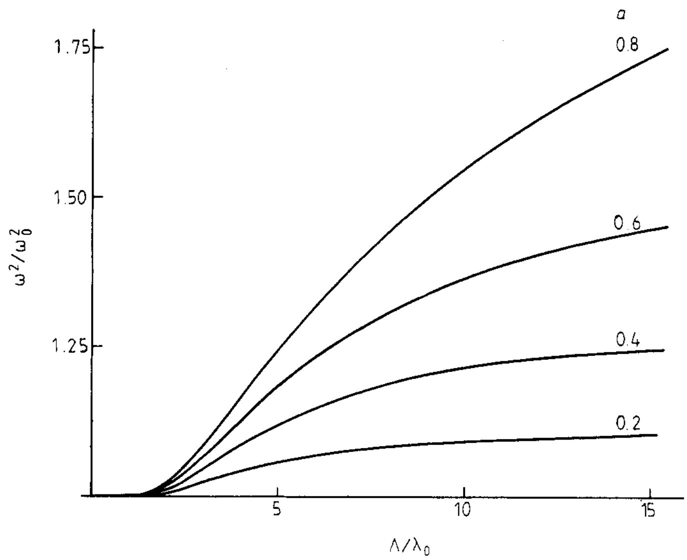

Figure 3. As figure 1, but the free energy F_s for the vortex line of high energy is given. The difference in F_s, compared to figure 1, gives the absolute height of the barrier against free vortex movement.

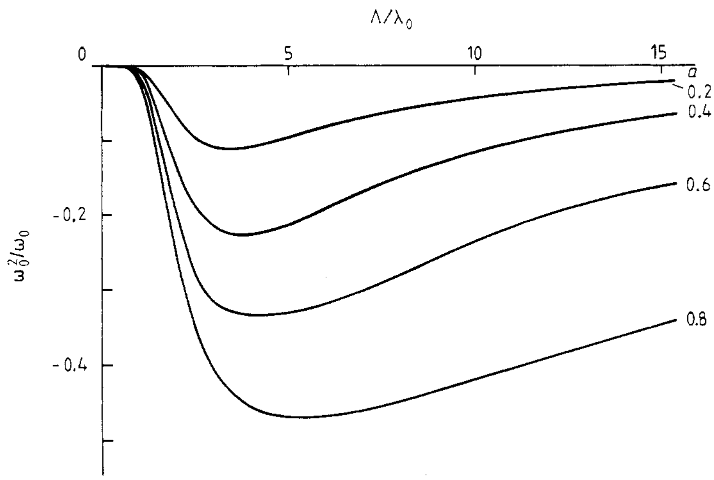

Figure 4. As figure 2, but the formal oscillation frequency ω is given for the vortex line of high energy. Note that here $\omega^2 < 0$, thus ω is imaginary and these vortex lines are unstable and rapidly begin to move along the junction.

external magnetic fields. Only for $H_0 > H_{c1b}$ it is possible to have a vortex line at the maximum of $\lambda_J(x)$. But note that $\omega^2 < 0$ for these vortex lines (figure 2) and thus they are unstable, since any small perturbation will first grow exponentially in time, resulting in a rapid movement of the vortex line along the junction. This is in contrast to the vortex lines of lower energy (figure 1), which have a real oscillation frequency and are stable against fluctuations.

In a simple analogy, we can think of the vortex line as a classical massive particle. If its mass is taken to be constant, then the pinning force which keeps the vortex line at its rest position has a force constant which is proportional to ω^2. Note that ω^2 and thus the pinning forces have their maxima around $\Lambda \simeq 3\lambda_0$ and that $\mathrm{d}F_s(\Lambda)/\mathrm{d}\Lambda$ has its maximum in the same region of Λ.

The free energy and oscillation frequency of vortex lines in a spatially modulated Josephson junction have been calculated systematically. For a finite range of the magnetic field H there could exist well defined vortex line lattices in the junction, which are stable against fluctuations and which could be manipulated by applying voltages across the junction and be used, for example, to store information. The strongest pinning forces have been found for a period of the modulation which is about three to four times the Josephson penetration depth (or coherence length). The pinning of vortex lines vanishes rapidly for smaller period lengths. Our results could also be applied qualitatively to other superconducting structures. In the high-T_c superconductors the spacing between the superconducting planes is probably too small to give efficient pinning of vortex lines and larger effects would be expected from the interaction of twinning planes with the superconductivity.

References

Ashcroft N W and Mermin N D 1976 *Solid State Physics* (New York: Holt, Rinehart and Winston) pp 133–5
Barone A and Paterno G 1982 *Physics and Applications of the Josephson Effect* (New York: Wiley) p 269
Dahlquist G and Björk Å 1974 *Numerical Methods* (Englewood Cliffs, NJ: Prentice Hall) p 359
Fetter A L and Stephen M J 1968 *Phys. Rev.* **168** 475
Kulić M L 1987 *Solid State Commun.* **63** 537
Kulik I O 1967 *Sov. Phys.–JETP* **24** 1307
Saint-James D, Sarma G and Thomas E J 1969 *Type-II Superconductivity* (Oxford: Pergamon)
Stampfli P and Rice T M 1986 *J. Phys. C: Solid State Phys.* **19** 6505–24